Markov Chain Exercises

SDMC - Gauss

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1. Which of these processes are Markov processes and which are not? For each part, if the problem describes a Markov process, draw a state diagram, write the transition matrix, and determine whether the process is regular. Otherwise, say why the process is not Markov.
   1. Math, Inc. is a large company that employs mathematicians. 5% of mathematicians not employed by Math, Inc. will join the company each year. 98% of the mathematicians employed by Math, Inc. who have worked there at least five years will choose to stay for another year, while the remaining 2% will quit. Only 85% of the newer employees (those who have worked at Math, Inc. for less than five years) will choose each year to remain with the company.

ANSWER: This is not a Markov process because it requires memory of how long an employee has been with the company. This means that the likelihood of an employee leaving or staying with the company cannot simply be determined based on whether the person is currently an employee.

* 1. Sometimes you bike to school, and other times you walk. You prefer to vary how you get to school, so if you used one method of transportation yesterday, there is a 10% chance that you will use the same method today. Otherwise, you will switch.

ANSWER: This is a Markov process because your decision of whether to walk or bike to school is based solely on how you got to school yesterday. Moreover, this is a regular Markov process because you can always transition from walking to biking, and vice versa. You’ll never get stuck doing one thing. The transition matrix is  or  , depending on how you set things up.

* 1. A particle moves on each of the eight vertices of a cube, where it is equally likely at each step to move to each of the adjacent vertices.

ANSWER: This is a Markov process, because where the particle ends up after the next step depends only on where it is currently located. Where the particle used to be several steps ago is completely irrelevant to where it will go next. This is also a regular Markov process because it is possible to get from any vertex on a cube to any other vertex on the cube by a sequence of steps. Your state diagram and transition matrix will depend on how you label the vertices of the cube. You should have an 8x8 matrix populated with 1/3’s and 0’s. Each column and each row should sum to 1.

* 1. You choose to move from one place to another by a series of jumps and somersaults. Your first move is a jump. After each jump, there is a 75% chance you will jump again, and a 25% chance you will switch to doing somersaults. You like to do somersaults, but if you do too many, you get dizzy. If you’ve already done 3 somersaults in a row, you switch to jumping with probability 1. If you’ve done less than 3 somersaults, then your next move will be a somersault with probability 0.06. That is, with probability 0.4, you switch back to jumping.

ANSWER: This is not a Markov process, because how likely you are to do another somersault depends on your recent history of moves, not just on your past move. For this to be a Markov process, you would need the probability of doing another somersault to be completely independent of your last three moves. It should only depend on your very last move because Markov processes have no memory.

1. We used the fact that xk=Pkx0. Prove formally that this is true for all positive integer values of k.

ANSWER: This fact is proven by mathematical induction. First, the base case (k=1):

x1=P1x0 because x1=Px0 by definition. Now assume that xk=Pkx0. Then xk+1=Pxk by definition. This means xk+1=P•Pkx0 by a substitution using the assumption that xk=Pkx0. Thus, by adding exponents, we get that xk+1=Pk+1x0. This proves the fact for all integer values of k ≥ 1.

1. One way of finding the steady-state vector is to find the eigenvectors of the transition matrix. Suppose an n-by-n transition matrix has eigenvectors v1, v2, … vn corresponding to distinct eigenvalues λ1, λ2, … λn. To find the steady-state vector, we need to find the inverse of the matrix of whose columns are v1, v2, … vn. Why is this matrix always invertible?

ANSWER: This is a linear algebra question. Eigenvectors associated with distinct eigenvalues are always linearly independent. This means the vectors v1, v2, … vn are linearly independent. Since there are n vectors, the n-by-n matrix with these vectors as columns has rank n. Any matrix with full rank is invertible.

1. Snakes and Ladders is one game that can be modeled by a Markov process. Think of another game that can be modeled by a Markov process and use what you learned to figure out the number of turns after which you can expect, with 95% probability, to complete the game.

ANSWER: Let me know if you think of a cool one!

1. Suppose the internet consisted of six web pages linked together in the following way:

page 1 links to page 2

page 1 links to page 6

page 2 links to page 1

page 2 links to page 6

page 3 links to page 1

page 3 links to page 5

page 5 links to page 1

page 5 links to page 3

page 5 links to page 6

Suppose that Google’s Page Rank Algorithm assumes that 88% of the time, a random surfer will click a link on the current page, and 12% of the time, they will type in a URL to get to any other page. How would the algorithm rank these pages?

ANSWER: The pages would be ranked in the following order: 1, 6, 2, 3, 5, 4 (or 1, 6, 2, 5, 3, 4, depending on how ties are resolved.)